Geodesic Ptolemy spaces and fixed point theory

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Abstract

We prove that geodesic Ptolemy spaces with a continuous midpoint map are strictly convex. Moreover, we show that geodesic Ptolemy spaces with a uniformly continuous midpoint map are reflexive and that in such a setting bounded sequences have unique asymptotic centers. These properties will then be applied to yield a series of fixed point results specific to CAT(0) spaces.

Key words: Ptolemy inequality, geodesic space, fixed point, nonexpansive mapping

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1. Introduction

In a metric space (X, d), the Ptolemy inequality says that

$$d(x, y)d(z, p) \leq d(x, z)d(y, p) + d(x, p)d(y, z) \text{ for every } x, y, z, p \in X.$$

A metric space where the Ptolemy inequality holds is called a Ptolemy metric space. It is shown in [16] that a normed space is an inner product space if and only if it is a Ptolemy space. Also, for each normed space $(X, \|\cdot\|)$, there exists a constant $1 \leq C \leq 2$ such that

$$\|x - y\|\|z - p\| \leq C\,(\|x - z\|\|y - p\| + \|x - p\|\|y - z\|) \text{ for every } x, y, z, p \in X,$$

(see for instance [14]). The smallest value of C such that the above inequality is satisfied is called the Ptolemy constant of the space X. The recent paper [12] examines the relation between the Ptolemy constant and the geometry of the space with application to the metric fixed point theory.

Moving into the metric setting, T. Foertsch and V. Schroeder prove in [8] that the Ptolemy inequality holds in the context of bounderies of CAT(-1) spaces endowed with an appropriate metric. This property will be used to study the relation between Gromov hyperbolic spaces and CAT(-1) spaces (see [8] for detalis).

CAT(0) spaces are Ptolemy spaces (see for instance [2] for a justification), but a geodesic Ptolemy space is not necessary CAT(0). In fact, T. Foertsch, A. Lytchak and V. Schroeder give in [7] an example of a

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geodesic Ptolemy space which is not even uniquely geodesic and thus not CAT(0). In the same paper it is shown that a proper geodesic Ptolemy space is uniquely geodesic where the properness condition may be replaced by the existence of a continuous midpoint map. Naturally, the authors raise the open question whether a proper geodesic Ptolemy space (or a geodesic Ptolemy space with a continuous midpoint map) is CAT(0).

It is easy to see that CAT(0) spaces are Busemannn convex but being Busemann convex is a weaker property than being CAT(0). However, in [7] a characterization of CAT(0) spaces is given in terms of the Busemann convexity and the Ptolemy inequality. Namely, it is shown that a metric space is CAT(0) if and only if it is Busemann convex and Ptolemy.

The purpose of this paper is to investigate geodesic Ptolemy spaces from the point of view of the fixed point theory. We prove that many known fixed point results for CAT(0) spaces can be stated in the context of a geodesic Ptolemy space with a uniformly continuous midpoint map. We do not have an answer to the question whether a geodesic Ptolemy space with a uniformly continuous midpoint map is CAT(0), but we provide a simple example which shows that having a uniformly continuous midpoint map is less restrictive than being Busemann convex and hence the characterization of CAT(0) spaces through the Busemannn convexity and the Ptolemy inequality cannot be applied to conclude that such a space is CAT(0). In section 3 we prove that a geodesic Ptolemy space with a continuous midpoint map is strictly convex. Further, we show that a geodesic Ptolemy space with a uniformly continuous midpoint map is reflexive and that in such a setting bounded sequences have a unique asymptotic center. This result will be used in section 4 to prove fixed point theorems in geodesic Ptolemy spaces with a uniformly continuous midpoint map starting from Kirk's fixed point theorem, continuing with generalized asymptotic pointwise contractions and nonexpansive mappings and ending with a fixed point result for multivalued mappings.

2. Preliminaries

Let (X, d) be a metric space. A *geodesic path* from x to y is a mapping $c : [0, l] \subseteq \mathbb{R} \to X$ with $c(0) = x, c(l) = y$ and $d\left(c(t), c(t')\right) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c\left([0, l]\right)$ of c forms a *geodesic segment* which joins x and y and is not necessarily unique. If no confusion arises, we will use $[x, y]$ to denote a geodesic segment joining x and y. (X, d) is a *(unique) geodesic space* if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, and we will write $z = (1 - t)x + ty$ for simplicity. A *subset K of X is convex* if it contains any geodesic segment that joins every two points of it. More details about geodesic metric spaces can be found in [1, 3].

The *metric* $d : X \times X \to \mathbb{R}$ is said to be *convex* if for any $x, y, z \in X$ one has

$$d(x, (1 - t)y + tz) \leq (1 - t)d(x, y) + td(x, z) \text{ for all } t \in [0, 1].$$

Let X be a geodesic Ptolemy space, $x, y, z \in X$ and $t \in [0,1]$. Applying the Ptolemy inequality,

$$\begin{aligned} d\left(x,(1-t)y+tz\right)d(y,z) &\leq d\left(x,y\right)d\left((1-t)y+tz,z\right)+d\left(x,z\right)d\left((1-t)y+tz,y\right) \\ &= (1-t)d\left(x,y\right)d\left(y,z\right)+td\left(x,z\right)d\left(y,z\right), \end{aligned}$$

which yields that the metric of X is convex.

The geodesic space (X,d) is *Busemann convex* if given any pair of geodesic paths $c_1 : [0,l_1] \to X$ and $c_2 : [0,l_2] \to X$ with $c_1(0) = c_2(0)$ one has

$$d(c_1(tl_1), c_2(tl_2)) \leq td(c_1(l_1), c_2(l_2)) \text{ for all } t \in [0,1].$$

Applying a simple reasoning we notice that in the definition of Busemann convexity we can renounce to the condition $c_1(0) = c_2(0)$. Then,

$$d(c_1(tl_1), c_2(tl_2)) \leq (1-t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)) \text{ for all } t \in [0,1].$$

It is also clear that in Busemann convex space the metric is convex.

We say that X admits a *continuous midpoint map* if there exists a map $m : X \times X \to X$ such that

$$d(x, m(x,y)) = d(y, m(x,y)) = \frac{d(x,y)}{2} \text{ for all } x, y \in X,$$

and for $x, y, x_n, y_n \in X$ where $n \in \mathbb{N}$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ we have that $\lim_{n\to\infty} d(m(x_n, y_n), m(x,y)) = 0$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ consists of three points x_1, x_2 and x_3 in X (the *vertices* of the triangle) and three geodesic segments corresponding to each pair of points (the *edges* of the triangle). For the geodesic traingle Δ=$\Delta(x_1, x_2, x_3)$, a *comparison triangle* is a triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d(x_i, x_j) = d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1,2,3\}$.

A geodesic triangle Δ satisfies the *$CAT(0)$ inequality* if for every comparison triangle $\bar{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x,y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the comparison points of x and y, i.e., if $x = (1-t)x_i + tx_j$ then $\bar{x} = (1-t)\bar{x}_i + t\bar{x}_j$.

A geodesic space is a *$CAT(0)$ space* if every geodesic triangle satisfies the CAT(0) inequality.

The geodesic space (X,d) is *reflexive* if every descending sequence of nonempty, bounded, closed and convex subsets of X has nonempty intersection. A simple example of a reflexive metric space is a reflexive Banach space. Other examples include complete CAT(0) spaces, complete uniformly convex metric spaces with a monotone or a lower semi-continuous from the right modulus of uniform convexity (see [5, 11]) and others.

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of a metric space X. For $x \in X$ set

$$r(x, (x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* of (x_n) in K is given by

$$r(K,(x_n)) = \inf_{x \in K} r(x,(x_n)),$$

and the *asymptotic center* of (x_n) in K is the possibly empty set

$$A(K,(x_n)) = \{x \in K : r(x,(x_n)) = r(K,(x_n))\}.$$

If K is a convex and closed subset of a complete CAT(0) space, $A(K,(x_n))$ is a singleton for every bounded sequence (x_n) (see for example [4]). Likewise, the same holds for complete uniformly convex metric spaces with a monotone (or lower semi-continuous from the right) modulus of uniform convexity (see [5] for details).

The 2-*dimensional sphere* $\mathbb{S}^2$ is the set $\left\{x \in \mathbb{R}^3 : (x \mid x) = 1\right\}$ where $(\cdot \mid \cdot)$ is the Euclidean scalar product. Define $d : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ by assigning to each $(x,y) \in \mathbb{S}^2 \times \mathbb{S}^2$ the unique number $d(x,y) \in [0,\pi]$ such that $\cos d(x,y) = (x \mid y)$. Then $(\mathbb{S}^2, d)$ is a metric space. Geodesics in $\mathbb{S}^2$ are arcs of great circles in $\mathbb{S}^2$. If $x, y \in \mathbb{S}^2$ with $d(x,y) < \pi$ then there is a unique geodesic joining x and y. Also, balls of radius smaller than $\frac{\pi}{2}$ are convex. The *spherical angle* between two geodesics starting at the same point $x \in \mathbb{S}^2$ and with the unit vectors u and v such that $(u \mid x) = (v \mid x) = 0$ is the unique number $\alpha \in [0,\pi]$ for which $\cos\alpha = (u \mid v)$. Having a spherical triangle with vertices $x, y, z \in \mathbb{S}^2$, the *vertex angle* at x is the spherical angle between the sides joining x to y and x to z respectively. The *spherical law of cosines* states that in a spherical triangle with vertices $x, y, z \in \mathbb{S}^2$ and γ the vertex angle at x we have

$$\cos d(y,z) = \cos d(x,y) \cos d(x,z) + \sin d(x,y) \sin d(x,z) \cos\gamma.$$

More about n-dimensional spheres can be found in [1].

Let X be a metric space, $K \subseteq X$ and $T : K \to K$. The *diameter* of K is $\operatorname{diam}(K) = \sup_{x,y \in K} d(x,y)$. Throughout this paper we will denote the fixed point set of T by $\operatorname{Fix}(T)$ (the same notation will also be used for multi-valued operators). The mapping T is called a *Picard operator* if it has a unique fixed point z and $(T^n(x))$ converges to z for each $x \in K$. We define the *orbit starting at* $x \in K$ by

$$O_T(x) = \left\{x, T(x), T^2(x), \ldots, T^n(x), \ldots\right\},$$

where $T^{n+1}(x) = T\left(T^n(x)\right)$ for $n \geq 0$ and $T^0(x) = x$. Denote also $O_T(x,y) = O_T(x) \cup O_T(y)$.

Following [9], we say that a family $\mathcal{F}$ of subsets of X defines a *convexity structure* on X if it contains the closed balls and is stable by intersection. A subset of X is *admissible* if it is a nonempty intersection of closed balls. The class of admissible subsets of X defines a convexity structure on X. A *convexity structure* $\mathcal{F}$ is called *compact* if any family $(A_\alpha)_{\alpha \in \Gamma}$ of elements of $\mathcal{F}$ has nonempty intersection provided $\bigcap_{\alpha \in F} A_\alpha \neq \emptyset$ for any finite subset $F \subseteq \Gamma$. Similarly, a *convexity structure* $\mathcal{F}$ is called *nested compact* (see [10]) if any descending family $(A_\alpha)_{\alpha \in \Gamma}$ of nonempty and bounded elements of $\mathcal{F}$ has nonempty intersection.

3. Properties of geodesic Ptolemy spaces

In [7] it is shown that a geodesic Ptolemy space does not even have to be a uniquely geodesic space. However, in that same work it is proved that these spaces can get very regular if certain properties are added. For instance, if continuity of a midpoint map is considered then every geodesic Ptolemy space is uniquely geodesic. Moreover, if the geodesic Ptolemy space is also Busemann convex, then it is as regular as a CAT(0) space. In this section we focus in this direction to obtain new results about the regularity of geodesic Ptolemy spaces when conditions on the midpoint map are considered. We begin with a result stating that geodesic Ptolemy spaces with a continuous midpoint map are strictly convex. The proof of this result has its roots in a construction used in the proof of [7, Theorem 1.2] where it is shown that a proper geodesic Ptolemy space is uniquely geodesic.

Theorem 3.1. *Let X be a geodesic Ptolemy space with a continuous midpoint map. Then X is strictly convex.*

Proof. Let $r > 0$ and $p, x, y \in X$ with $d(p, x) \le r$, $d(p, y) \le r$, $d(x, y) > 0$. By the convexity of the metric we know that $d\left(p, \frac{1}{2}x + \frac{1}{2}y\right) \le \frac{1}{2}\left(d(p, x) + d(p, y)\right)$. If $d(p, x) < r$ or $d(p, y) < r$ it follows that $d\left(p, \frac{1}{2}x + \frac{1}{2}y\right) < r$ and we are done. Otherwise, suppose $d(p, x) = r$, $d(p, y) = r$ and $d\left(p, \frac{1}{2}x + \frac{1}{2}y\right) = r$. Let $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < r$. Henceforth we will assume that $n \ge n_0$. Take $x_n \in [p, x]$ and $y_n \in [p, y]$ with $d(x_n, x) = d(y_n, y) = \frac{1}{n}$. This implies that $d(p, x_n) = d(p, y_n) = r - \frac{1}{n}$.
Let $l = d(x, y)$, $l_n = d(x_n, y_n)$, $a_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x\right)$, $a'_n = d\left(\frac{1}{2}x + \frac{1}{2}y, x_n\right)$, $b_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, y\right)$, $b'_n = d\left(\frac{1}{2}x + \frac{1}{2}y, y_n\right)$ and $m_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x + \frac{1}{2}y\right)$. Since X has a continuous midpoint map it follows that $\lim_{n\to\infty} m_n = 0$.
Then,

$$l \le a_n + b_n. \tag{1}$$

By the Ptolemy inequality,

$$d\left(p, \frac{1}{2}x + \frac{1}{2}y\right) d(x_n, y_n) \le d(p, x_n) d\left(\frac{1}{2}x + \frac{1}{2}y, y_n\right) + d(p, y_n) d\left(\frac{1}{2}x + \frac{1}{2}y, x_n\right)$$

and so

$$d\left(\frac{1}{2}x + \frac{1}{2}y, y_n\right) + d\left(\frac{1}{2}x + \frac{1}{2}y, x_n\right) \ge \frac{r}{r - \frac{1}{n}} d(x_n, y_n),$$

which means that

$$a'_n + b'_n \ge \frac{r}{r - \frac{1}{n}} l_n. \tag{2}$$

Applying again the Ptolemy inequality we have that

$$d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x\right) d\left(\frac{1}{2}x + \frac{1}{2}y, x_n\right) \le \frac{1}{n} d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x + \frac{1}{2}y\right) + \frac{1}{4} d(x, y) d(x_n, y_n),$$

that is,

$$a_n a_n' \le \frac{1}{n} m_n + \frac{1}{4} l l_n. \tag{3}$$

Likewise,

$$b_n b_n' \le \frac{1}{n} m_n + \frac{1}{4} l l_n. \tag{4}$$

We also know that

$$d(x,y) \le d(x,x_n) + d(x_n,y_n) + d(y_n,y) = d(x_n,y_n) \text{ and } d(x,y) \ge d(x_n,y_n) - d(x,x_n) - d(y_n,y),$$

so $|l_n - l| \le \frac{2}{n}$. Since

$$d\left(\frac{1}{2}x + \frac{1}{2}y, x_n\right) \le d\left(\frac{1}{2}x + \frac{1}{2}y, x\right) + d(x,x_n) \text{ and } d\left(\frac{1}{2}x + \frac{1}{2}y, x_n\right) \ge d\left(\frac{1}{2}x + \frac{1}{2}y, x\right) - d(x,x_n),$$

it follows that $\left|a_n' - \frac{l}{2}\right| \le \frac{1}{n}$. Similarly, because

$$d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x\right) \le d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x_n\right) + d(x_n,x) \text{ and } d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x\right) \ge d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x_n\right) - d(x_n,x)$$

it is clear that $\left|a_n - \frac{l_n}{2}\right| \le \frac{1}{n}$ yielding $\left|a_n - \frac{l}{2}\right| \le \frac{2}{n}$. In the same way we obtain that $\left|b_n' - \frac{l}{2}\right| \le \frac{1}{n}$ and $\left|b_n - \frac{l}{2}\right| \le \frac{2}{n}$.

Now we can write $l_n = l + c_n \frac{1}{n}$ where $c_n \in [-2,2]$. We may assume that (c_n) is convergent (otherwise choose a convergence subsequence). Let $c = \lim_{n\to\infty} c_n$ and $S = \{(\lambda_n) \subseteq \mathbb{R} : \lim_{n\to\infty} n\lambda_n = 0\}$. Then $l_n = l + \frac{c}{n} + \frac{c_n - c}{n} = l + \frac{c}{n} + \gamma_n$, where $(\gamma_n) \in S$. Applying the same reasoning, there exist $a, a', b, b' \in [-2,2]$ and $(\alpha_n), (\alpha_n'), (\beta_n), (\beta_n') \in S$ such that $a_n = \frac{l}{2} + \frac{a}{n} + \alpha_n$, $a_n' = \frac{l}{2} + \frac{a'}{n} + \alpha_n'$, $b_n = \frac{l}{2} + \frac{b}{n} + \beta_n$ and $b_n' = \frac{l}{2} + \frac{b'}{n} + \beta_n'$. Using (1) we immediately obtain that $a + b \ge 0$. From (2) it follows that

$$l + \frac{a' + b'}{n} + \alpha_n' + \beta_n' \ge l + \frac{c}{n} + \gamma_n + \frac{1}{nr - 1}\left(l + \frac{c}{n} + \gamma_n\right),$$

which yields that

$$a' + b' + n(\alpha_n' + \beta_n') \ge c + n\gamma_n + \frac{1}{r - 1/n}\left(l + \frac{c}{n} + \gamma_n\right).$$

Letting $n \to \infty$ we get that $a' + b' \ge c + \frac{l}{r}$.

Based on (3) and (4) we have

$$\left(\frac{l}{2} + \frac{a}{n} + \alpha_n\right)\left(\frac{l}{2} + \frac{a'}{n} + \alpha_n'\right) + \left(\frac{l}{2} + \frac{b}{n} + \beta_n\right)\left(\frac{l}{2} + \frac{b'}{n} + \beta_n'\right) \le \frac{l}{2}\left(l + \frac{c}{n} + \gamma_n\right) + \frac{2}{n} m_n,$$

which means that

$$\frac{l}{2}(a + a' + b + b') \le \frac{l}{2} c + n\psi_n, \tag{5}$$

$$\psi_n = \frac{l}{2}\left(\gamma_n - \alpha_n - \alpha_n' - \beta_n - \beta_n'\right) + \frac{1}{n}\left(2m_n - a'\alpha_n - a\alpha_n' - b'\beta_n - b\beta_n'\right) - \alpha_n\alpha_n' - \beta_n\beta_n'.$$

Since $\lim_{n\to\infty} m_n = 0$ we get $(\psi_n) \in S$.

Letting $n \to \infty$ in (5) we obtain that $a + a' + b + b' \le c$. But this yields that $l = 0$, so $d(x,y) = 0$ which is a contradiction. This ends the proof. $\square$

To prove the reflexivity of geodesic Ptolemy spaces we remind the following notion.

Definition 3.2. *Let X be a geodesic space. We say that X admits a uniformly continuous midpoint map if there exists a map $m : X \times X \to X$ such that*

$$d(x, m(x,y)) = d(y, m(x,y)) = \frac{d(x,y)}{2} \text{ for all } x, y \in X,$$

and for $n \in \mathbb{N}$ and $x_n, x_n', y_n, y_n' \in X$ with $\lim_{n\to\infty} d(x_n, x_n') = 0$ and $\lim_{n\to\infty} d(y_n, y_n') = 0$ we have that $\lim_{n\to\infty} d(m(x_n, y_n), m(x_n', y_n')) = 0$.

Clearly, every Busemann convex geodesic space admits a uniformly continuous midpoint map. The following example shows however that there exist spaces with a uniformly continuous midpoint map, but without being Busemann convex.

Example 3.3. *Let X be the positive octant of the spherical space $(\mathbb{S}^2, d)$. Then X is not Busemann convex, but admits a uniformly continuous midpoint map.*

Proof. Notice that X is uniquely geodesic and $\operatorname{diam}(X) = \frac{\pi}{2}$. To prove that X is not Busemann convex, let $x, y, z \in X$ be vertices of a spherical triangle in X such that $d(x,y) = \frac{\pi}{3}$, $d(x,z) = \frac{\pi}{3}$ and $d(y,z) = \frac{\pi}{4}$. Let $m_1 = \frac{1}{2}x + \frac{1}{2}y$ and $m_2 = \frac{1}{2}x + \frac{1}{2}z$. Applying the spherical law of cosines in the spherical triangles with vertices x, y, z and x, m_1, m_2 respectively,

$$\begin{aligned}\cos d(m_1, m_2) &= \frac{1 + \cos d(x,y) + \cos d(x,z) + \cos d(y,z)}{4 \cos \frac{d(x,y)}{2} \cos \frac{d(x,z)}{2}} = \frac{4+\sqrt{2}}{6} \\ &< \frac{\sqrt{2+\sqrt{2}}}{2} = \cos\frac{\pi}{8} = \cos\frac{d(y,z)}{2}.\end{aligned}$$

Hence, $d(m_1, m_2) > \frac{d(y,z)}{2}$ which proves that X is not Busemann convex.
Now, for $n \in \mathbb{N}$, suppose $x_n, x_n', y_n, y_n' \in X$ with $\lim_{n\to\infty} d(x_n, x_n') = 0$ and $\lim_{n\to\infty} d(y_n, y_n') = 0$. Denote

$$m_n = \frac{1}{2}x_n + \frac{1}{2}y_n, \quad m_n' = \frac{1}{2}x_n' + \frac{1}{2}y_n' \quad \text{and} \quad m_n'' = \frac{1}{2}x_n' + \frac{1}{2}y_n.$$

Since $d(m_n, m_n') \le d(m_n, m_n'') + d(m_n'', m_n')$, it is enough to prove that $\lim_{n\to\infty} d(m_n, m_n'') = 0$ and $\lim_{n\to\infty} d(m_n', m_n'') = 0$. Knowing that

$$\begin{aligned}\cos d(m_n, m_n'') &= \frac{2\cos^2 d(x_n, m_n) + 2\cos^2 d(x_n', m_n'') + \cos d(x_n, x_n') - 1}{4 \cos d(x_n, m_n) \cos d(x_n', m_n'')} \\ &= \frac{2\left(\cos d(x_n, m_n) - \cos d(x_n', m_n'')\right)^2 + \cos d(x_n, x_n') - 1}{4 \cos d(x_n, m_n) \cos d(x_n', m_n'')} + 1 \\ &\ge \frac{\cos d(x_n, x_n') - 1}{4 \cos d(x_n, m_n) \cos d(x_n', m_n'')} + 1,\end{aligned}$$

and letting $n \to \infty$ we obtain that $\lim_{n\to\infty} d(m_n, m_n'') = 0$. Similarly, $\lim_{n\to\infty} d(m_n', m_n'') = 0$. $\square$

Since every geodesic Ptolemy space with a continuous midpoint map is uniquely geodesic it immediately follows that every geodesic Ptolemy space with a uniformly continuous midpoint map is also uniquely geodesic. We prove next the reflexivity of geodesic Ptolemy spaces with a uniformly continuous midpoint map. This result will be the key tool in proving Theorem 3.6.

Theorem 3.4. *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map. Then X is reflexive.*

Proof. Let (C_n) be a decreasing sequence of nonempty, bounded, closed and convex subsets of X. We may suppose we can find $x \in X$ and $n_0 \in \mathbb{N}$ such that $x \notin C_n$ for every $n \geq n_0$. Let $r_n = \operatorname{dist}(x, C_n)$. For $n \geq n_0$, $r_n > 0$. It is also obvious that $r_n \leq r_{n+1}$ and (r_n) is a bounded sequence. Hence, (r_n) is convergent to some $r > 0$.

Let us first suppose that $r_n < r$ for every $n \in \mathbb{N}$. Consider the sets $A_n = C_n \cap \widetilde{B}(x, r)$. Then (A_n) is a decreasing sequence of nonempty and closed subsets of X. The sequence $(\operatorname{diam}(A_n))$ is decreasing and bounded so it converges to some $d \geq 0$. Suppose $d > 0$.

Take $n_1 \in \mathbb{N}$ such that $r_n > 2(r - r_n)$ for $n \geq n_1$ (this is possible because otherwise $r = 0$ which is false). For $n \geq n_1$, pick $x_n, y_n \in A_n$ such that $d(x_n, y_n) > d/2$. Let $x_n' \in [x, x_n]$ and $y_n' \in [x, y_n]$ such that $d(x_n, x_n') = d(y_n, y_n') = 2(r - r_n)$. Since $x_n, y_n \in A_n$ it follows that $d(x, x_n) \leq r$ and $d(x, y_n) \leq r$. Thus, $d(x, x_n') \leq 2r_n - r$ and $d(x, y_n') \leq 2r_n - r$.

Let $l_n = d(x_n, y_n)$, $l_n' = d(x_n', y_n')$, $a_n = d\left(\frac{1}{2}x_n' + \frac{1}{2}y_n', x_n\right)$, $a_n' = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x_n'\right)$, $b_n = d\left(\frac{1}{2}x_n' + \frac{1}{2}y_n', y_n\right)$, $b_n' = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, y_n'\right)$ and $m_n = d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x_n' + \frac{1}{2}y_n'\right)$. Since X has a uniformly continuous midpoint map it follows that $\lim_{n\to\infty} m_n = 0$.

Clearly,

$$l_n \leq a_n + b_n. \tag{6}$$

By the Ptolemy inequality,

$$d\left(x, \frac{1}{2}x_n + \frac{1}{2}y_n\right) d(x_n', y_n') \leq d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x_n'\right) d(x, y_n') + d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, y_n'\right) d(x, x_n').$$

Since the set C_n is convex this implies that

$$d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x_n'\right) + d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, y_n'\right) \geq \frac{r_n}{2r_n - r} d(x_n', y_n'),$$

that is,

$$a_n' + b_n' \geq \frac{r_n}{2r_n - r} l_n'. \tag{7}$$

Applying again the Ptolemy inequality we have that

$$d\left(\frac{1}{2}x_n' + \frac{1}{2}y_n', x_n\right) d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, x_n'\right) \leq 2(r - r_n) d\left(\frac{1}{2}x_n + \frac{1}{2}y_n, \frac{1}{2}x_n' + \frac{1}{2}y_n'\right) + \frac{1}{4} d(x_n, y_n) d(x_n', y_n').$$

Consequently,

$$a_n a'_n \leq 2(r-r_n)m_n + \frac{1}{4} l_n l'_n. \tag{8}$$

Likewise,

$$b_n b'_n \leq 2(r-r_n)m_n + \frac{1}{4} l_n l'_n. \tag{9}$$

Similarly as in the proof of Theorem 3.1 using appropriate triangle inequalities we have that $|l_n - l'_n| \leq 4(r-r_n)$, $\left|a'_n - \frac{l_n}{2}\right| \leq 2(r-r_n)$, $\left|a_n - \frac{l_n}{2}\right| \leq 4(r-r_n)$, $\left|b'_n - \frac{l_n}{2}\right| \leq 2(r-r_n)$ and $\left|b_n - \frac{l_n}{2}\right| \leq 4(r-r_n)$.
Let $S = \left\{(\lambda_n) \subseteq \mathbb{R} : \lim_{n\to\infty} \frac{\lambda_n}{r-r_n} = 0\right\}$. Applying again the same reasoning as in the proof of Theorem 3.1 we can conclude that there exist $a, a', b, b', c \in [-4, 4]$ and $(\alpha_n), (\alpha'_n), (\beta_n), (\beta'_n), (\gamma_n) \in S$ such that $a_n = \frac{l_n}{2} + a(r-r_n) + \alpha_n$, $a'_n = \frac{l_n}{2} + a'(r-r_n) + \alpha'_n$, $b_n = \frac{l_n}{2} + b(r-r_n) + \beta'_n$, $b'_n = \frac{l_n}{2} + b'(r-r_n) + \beta'_n$ and $l'_n = l_n + c(r-r_n) + \gamma_n$.
From (6) it follows that $a + b \geq 0$. Using (7) we have

$$l_n + (a'+b')(r-r_n) + \alpha'_n + \beta'_n \geq l_n + c(r-r_n) + \gamma_n + \frac{r-r_n}{2r_n - r}\left(l_n + c(r-r_n) + \gamma_n\right),$$

which yields that

$$a' + b' + \frac{\alpha'_n + \beta'_n}{r-r_n} \geq c + \frac{\gamma_n}{r-r_n} + \frac{1}{2r_n - r}\left(\frac{d}{2} + c(r-r_n) + \gamma_n\right).$$

Taking the limit when $n \to \infty$ we obtain that $a' + b' \geq c + \frac{d}{2r}$.
Applying (8) and (9) we have that

$$\left(\frac{l_n}{2} + a(r-r_n) + \alpha_n\right)\left(\frac{l_n}{2} + a'(r-r_n) + \alpha'_n\right) + \left(\frac{l_n}{2} + b(r-r_n) + \beta_n\right)\left(\frac{l_n}{2} + b'(r-r_n) + \beta'_n\right) \leq \frac{l_n}{2}\left(l_n + c(r-r_n) + \gamma_n\right) + 4(r-r_n)m_n,$$

which means that

$$\frac{l_n}{2}(a + a' + b + b') \leq \frac{l_n}{2} c + \frac{\psi_n}{r-r_n}, \tag{10}$$

where

$$\psi_n = \frac{l_n}{2}\left(\gamma_n - \alpha_n - \alpha'_n - \beta_n - \beta'_n\right) + (r-r_n)\left(4m_n - a'\alpha_n - a\alpha'_n - b'\beta_n - b\beta'_n\right) - \alpha_n\alpha'_n - \beta_n\beta'_n.$$

Since $\lim_{n\to\infty} m_n = 0$ we get $(\psi_n) \in S$.
Taking in (10) the superior limit when $n \to \infty$ and knowing $l_n > \frac{d}{2}$ it follows that $a + a' + b + b' \leq c$. But this implies $d = 0$ which is a contradiction. Hence, $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$. Since X is complete, we obtain that $\bigcap_n A_n \neq \emptyset$ and so $\bigcap_n C_n \neq \emptyset$.
Now suppose there exists $n_2 \in \mathbb{N}$ such that $r_n = r$ and $r > \frac{2}{n}$ for each $n \geq n_2$. Take $A_n = C_n \cap \widetilde{B}(x, r + \frac{1}{n})$. In order to complete the proof we need to show that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$. Assume this limit is $d > 0$. For $n \geq n_2$, let $x_n, y_n \in A_n$ such that $d(x_n, y_n) > d/2$. Let $x'_n \in [x, x_n]$ and $y'_n \in [x, y_n]$ such that

$d(x_n, x'_n) = d(y_n, y'_n) = \frac{2}{n}$. Since $x_n, y_n \in A_n$ it follows that $d(x, x'_n) \le r - \frac{1}{n}$ and $d(x, y'_n) \le r - \frac{1}{n}$. Using the same notations as above and a similar argument one can show that

$$l_n \le a_n + b_n, \quad a'_n + b'_n \ge \frac{r}{r - 1/n} l'_n \quad \text{and} \quad a_n a'_n + b_n b'_n \le \frac{1}{2} l_n l'_n + \frac{4}{n} m_n.$$

It can be proved that there exist $a, a', b, b, c \in [-4, 4]$ and $(\alpha_n), (\alpha'_n), (\beta_n), (\beta'_n), (\gamma_n) \in S'$, where $S' = \{(\lambda_n) \subseteq \mathbb{R} : \lim_{n\to\infty} n\lambda_n = 0\}$, such that $a_n = \frac{l_n}{2} + \frac{a}{n} + \alpha_n$, $a'_n = \frac{l_n}{2} + \frac{a'}{n} + \alpha'_n$, $b_n = \frac{l_n}{2} + \frac{b}{n} + \beta_n$, $b'_n = \frac{l_n}{2} + \frac{b'}{n} + \beta'_n$ and $l'_n = l_n + \frac{c}{n} + \gamma_n$. As above, it follows that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$ and we are done. $\square$

Remark 3.5. *So far reflexivity of geodesic metric spaces has been proved only for uniformly convex spaces (see [5, 11]). Therefore, a natural question at this point is to know if geodesic Ptolemy spaces with a uniformly continuous mipoint map are uniformly convex. Notice that this question lies somewhere in between what it is known and the open question raised in [7] of whether such spaces are CAT(0).*

The following theorem lies at the heart of proving fixed point results in geodesic Ptolemy spaces.

Theorem 3.6. *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map, $(x_n)_{n\in\mathbb{N}} \subseteq X$ a bounded sequence and $K \subseteq X$ closed and convex. Then $(x_n)_{n\in\mathbb{N}}$ has a unique asymptotic center in K.*

Proof. Let $r = r(K, (x_n))$ and $k \in \mathbb{N}$. Then there exists $x \in K$ such that $r(x, (x_n)) < r + \frac{1}{k}$. Hence, for n sufficiently large, $d(x, x_n) < r + \frac{1}{k}$, that is, $x \in B(x_n, r + 1/k)$. Consider the set

$$C_k = \bigcup_{j\ge 1} \left(\bigcap_{i\ge j} B(x_i, r + 1/k) \cap K \right).$$

Since the metric of X is convex, C_k will be convex. The set $\overline{C_k}$ will also be convex because X is complete and has a continuous midpoint map. Hence, the sequence $(\overline{C_k})$ is a descending sequence of nonempty, bounded, closed and convex sets and so, by Theorem 3.4, there exists $u \in K$ such that $u \in \bigcap_k C_k$. This yields that $r(u, (x_n)) = r$.

Suppose there exists $v \in K$, $v \ne u$ such that $r(v, (x_n)) = r$. Clearly, $r > 0$ because otherwise $v = u$. For $n \in \mathbb{N}$, let $r_n = d\left(\frac{1}{2}u + \frac{1}{2}v, x_n\right)$. The convexity of the metric implies that $r\left(\frac{1}{2}u + \frac{1}{2}v, (x_n)\right) = r$. Thus, there exists $(x_k) \subseteq (x_n)$ with $\lim_{k\to\infty} r_k = r$.

Pick $p_0 \in \mathbb{N}$ such that $\frac{4}{p_0} < \frac{r}{2}$ and let $p \ge p_0$. It is clear that there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$, $d(u, x_k) < r + \frac{1}{p}$ and $d(v, x_k) < r + \frac{1}{p}$. Likewise, there exists $k_1 \in \mathbb{N}$ such that for $k \ge k_1$, $r - \frac{1}{p} < r_k$. At the same time, there exists $k_2 \in \mathbb{N}$ such that for $k \ge k_2$, $d(u, x_k) > \frac{3}{p}$ and $d(v, x_k) > \frac{3}{p}$. Otherwise, there exists a subsequence $(x_s) \subseteq (x_k)$ such that $d(u, x_s) \le \frac{3}{p}$ or $d(v, x_s) \le \frac{3}{p}$. Then, for s sufficiently large,

$$r_s = d\left(\frac{1}{2}u + \frac{1}{2}v, x_s\right) \le \frac{1}{2}\left(d(u, x_s) + d(v, x_s)\right) < \frac{1}{2}\left(\frac{3}{p} + r + \frac{1}{p}\right) < \frac{3}{4}r.$$

Letting $s \to \infty$ we have that $r = 0$ which is false.
Thus, for $p \geq p_0$ there exists $k = k(p)$ such that

$$\frac{3}{p} < d(u, x_k) < r + \frac{1}{p}, \quad \frac{3}{p} < d(v, x_k) < r + \frac{1}{p} \quad \text{and} \quad r - \frac{1}{p} < r_k.$$

Take $u_p \in [u, x_k]$ and $v_p \in [v, x_k]$ such that $d(u, u_p) = d(v, v_p) = \frac{3}{p}$.
Let $l = d(u, v)$, $l_p = d(u_p, v_p)$, $a_p = d\left(\frac{1}{2}u_p + \frac{1}{2}v_p, u\right)$, $a'_p = d\left(\frac{1}{2}u + \frac{1}{2}v, u_p\right)$, $b_p = d\left(\frac{1}{2}u_p + \frac{1}{2}v_p, v\right)$, $b'_p = d\left(\frac{1}{2}u + \frac{1}{2}v, v_p\right)$ and $m_p = d\left(\frac{1}{2}u_p + \frac{1}{2}v_p, \frac{1}{2}u + \frac{1}{2}v\right)$. Since X has a continuous midpoint map it follows that $\lim_{p\to\infty} m_p = 0$.
Similarly as in the proofs of Theorem 3.1 and 3.4 one can show that

$$l \leq a_p + b_p, \quad a'_p + b'_p \geq \frac{r - 1/p}{r - 2/p} l_p \quad \text{and} \quad a_p a'_p + b_p b'_p \leq \frac{6}{p} m_p + \frac{1}{2} l l_p.$$

It can be also proved that there exist $a, a', b, b, c \in [-6, 6]$ and $(\alpha_p), (\alpha'_p), (\beta_p), (\beta'_p), (\gamma_p) \in S$, where $S = \{(\lambda_p) \subseteq \mathbb{R} : \lim_{p\to\infty} p\lambda_p = 0\}$, such that $a_p = \frac{l}{2} + \frac{a}{p} + \alpha_p$, $a'_p = \frac{l}{2} + \frac{a'}{p} + \alpha'_p$, $b_p = \frac{l}{2} + \frac{b}{p} + \beta_p$, $b'_p = \frac{l}{2} + \frac{b'}{p} + \beta'_p$ and $l_p = l + \frac{c}{p} + \gamma_p$. As in the above proofs it follows that $l = 0$, that is, $u = v$ which is a contradiction. Hence, the proof is complete. □

4. Fixed points in geodesic Ptolemy spaces with a uniformly continuous midpoint map

The properties of geodesic Ptloemy spaces established in section 3, especially Theorem 3.6, allow us to prove a large class of fixed point results in this framework. In this section we mention some of these results with the remark that most fixed point results whose proofs rely mainly on the uniqueness of asymptotic centers and the convexity of the metric can be transposed into this setting. We begin by giving Kirk's fixed point theorem in geodesic Ptolemy spaces.

Theorem 4.1. *Let X be a complete Ptolemy geodesic space with a uniformly continuous midpoint map and $K \subseteq X$ nonempty, bounded, closed and convex. Suppose $T : K \to K$ is a nonexpansive mapping. Then Fix(T) is nonempty, closed and convex.*

Proof. Let $x \in K$ and $x_n = T^n(x)$ for $n \in \mathbb{N}$. By Theorem 3.6, the sequence (x_n) has a unique asymptotic center denoted by u. Using the fact that T is nonexpansive we have that $r\left(T(u), (x_n)\right) \leq r\left(u, (x_n)\right)$ which implies that $T(u) = u$.
Since T is continuous, it follows that $\mathrm{Fix}(T)$ is closed. Now let $u, v \in \mathrm{Fix}(T)$ and $m = (1-\alpha)u + \alpha v$ for some $\alpha \in [0, 1]$. Then, $d(T(m), u) = d(T(m), T(u)) \leq d(m, u) = \alpha d(u, v)$. Likewise, $d(T(m), v) \leq (1-\alpha) d(u, v)$. Since by [7, Theorem 1.2] X is uniquely geodesic we obtain that $T(m) = m$. This completes the proof. □

We state next two theorems for some generalized nonexpansive mappings that were initially given in the CAT(0) setting in [13, Theorems 5.1, 5.3]. We omit the proofs because they basically follow the same patterns, but we must remark that for the proof of the convexity of the fixed point set, the (CN) inequality will we replaced with a similar argument as in the proof of Theorem 4.1.

Theorem 4.2. *Let X be a bounded complete Ptolemy geodesic space with a uniformly continuous midpoint map and let $T : X \to X$ be such that for every $x, y \in X$,*

$$d\left(T(x), T(y)\right) \leq \sup_{z \in O_T(y)} d(x, z).$$

Then $Fix(T)$ is nonempty, closed and convex.

Theorem 4.3. *Let X be a bounded complete Ptolemy geodesic space with a uniformly continuous midpoint map and let $T : X \to X$ be such that for every $x, y \in X$,*

$$d\left(T(x), T(y)\right) \leq \operatorname{diam}\left(\{x\} \cup O_T(y)\right),$$

and

$$d\left(T(x), T(y)\right) \leq \sup_{z \in O_T(y)} d(x, z) + \sup_{k,p \in \mathbb{N}} \left(\operatorname{diam}\left(\left\{T^k(x)\right\} \cup O_T\left(T^{k+p}(y)\right)\right) - \operatorname{diam} O_T\left(T^{k+p}(y)\right)\right).$$

Then $Fix(T)$ is nonempty, closed and convex.

In the sequel we formulate Theorems 3.2 and 4.2 of [13] in the context of Ptolemy geodesic spaces. These results were proved in bounded metric spaces where the class of admissible subsets defines a compact convexity structure. However, as remarked in [5], it suffices that the class of admissible subsets is nested compact. By Theorem 3.4, it is immediate that in the context of a complete Ptolemy geodesic space with a uniformly continuous midpoint map, the class of admissible subsets is nested compact. Hence, we can give the two next results.

Theorem 4.4. *Let X be a bounded complete Ptolemy geodesic space with a uniformly continuous midpoint map and $T : X \to X$ for which there exists $\alpha : X \to [0, 1)$ such that*

$$d\left(T(x), T(y)\right) \leq \alpha(x) \sup_{z \in O_T(y)} d(x, z) \text{ for every } x, y \in X.$$

Then T is a Picard operator.

Theorem 4.5. *Let X be a bounded complete Ptolemy geodesic space with a uniformly continuous midpoint map and $T : X \to X$ an orbitally continuous mapping. Also, for $n \in \mathbb{N}$, let $\alpha_n : X \to \mathbb{R}_+$ for which there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \leq k$. If for each $n \in \mathbb{N}$,*

$$d\left(T^n(x), T^n(y)\right) \leq \alpha_n(x) \operatorname{diam} O_T(x, y) \text{ for every } x, y \in X,$$

then T is a Picard operator.

The following is an analogue of [13, Theorem 3.4] in the setting of Ptolemy geodesic spaces. Because the proof is short, we will briefly sketch it.

Theorem 4.6. *Let X be a bounded complete Ptolemy geodesic space with a uniformly continuous midpoint map and $T : X \to X$ such that*

$$d\left(T^n(x), T^n(y)\right) \leq \alpha_n(x) \sup_{z \in O_T(y)} d(x,z) \text{ for every } x, y \in X,$$

where for each $n \in \mathbb{N}, \alpha_n : X \to \mathbb{R}_+$ and the sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges pointwise to a function $\alpha : X \to [0,1)$. Then T is a Picard operator.

Proof. Assume T has two fixed points $x, y \in X, x \neq y$. Then for each $n \in \mathbb{N}$, $d(x,y) \leq \alpha_n(x) d(x,y)$. When $n \to \infty$ we obtain $\alpha(x) \geq 1$ which is false. Hence, T has at most one fixed point.
Let $x \in X$ and $x_n = T^n(x)$ for $n \in \mathbb{N}$. By Theorem 3.6, (x_n) has a unique asymptotic center denoted by u. For $p \in \mathbb{N}$,

$$r\left(u,(x_n)\right) \leq r\left(T^p(u),(x_n)\right) \leq \alpha_p(u) r\left(u,(x_n)\right).$$

Letting above $p \to \infty$ yields that (x_n) converges to u which will be the unique fixed point of T because based on the above inequality, $r\left(T(u),(x_n)\right) = 0$. Thus, all the Picard iterates will converge to u. □

As remarked in [6], the following theorem proved in [15] in the setting of a complete CAT(0) space (see also [17]) holds in more general contexts. We formulate this result in the framework of geodesic Ptolemy spaces with a uniformly continuous midpoint map since the proof only requires the uniqueness of asymptotic centers and the convexity of the metric, conditions which are satisfied in such a setting.

Theorem 4.7. *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map and suppose $K \in P_{b,cl,cv}(X)$. If $T : K \to K$ is such that*

$$\frac{1}{2} d(x, T(x)) \leq d(x,y) \Longrightarrow d(T(x), T(y)) \leq d(x,y) \text{ for all } x, y \in K,$$

then $\mathrm{Fix}(T)$ is nonempty, closed and convex.

We conclude this section by giving an example of a fixed point theorem for multi-valued mappings in geodesic Ptolemy spaces. We do not include the proof since it is an easy adaptation of the one given in [6, Theorem 3.7].

Theorem 4.8. *Let X be a complete geodesic Ptolemy space with a uniformly continuous midpoint map and $K \in P_{b,cl,cv}(X)$. Suppose $T : K \to P_{cp}(K)$ is such that for each $x, y \in K$ and $u_x \in T(x)$ with $\frac{1}{2} d(x, u_x) \leq d(x,y)$, there exists $u_y \in T(y)$ for which $d(u_x, u_y) \leq d(x,y)$. Then $\mathrm{Fix}(T) \neq \emptyset$.*

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